LATTICE FIELD THEORY 3

Strong Coupling & Confinement

Lattice Fermions

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Wilson Action

Last time we ended up with the Wilson plaquette action

$$S = \frac{\beta}{2N} \sum_{x,\mu,\nu} P_{\mu\nu}(x).$$

It is not too hard (and therefore an exercise) to show that

$$S \to \frac{1}{4g_0^2} \int d^4x \, (F^a_{\mu\nu})^2, \qquad \frac{1}{g_0^2} = \frac{\beta}{2N}$$

as $a \to 0$, β fixed.

Then the functional integral is

$$\langle \bullet \rangle = \frac{1}{Z} \int \mathcal{D}U \, \bullet \, e^{-S}, \qquad \mathcal{D}U = \prod_{x,\mu} dU_\mu(x)$$

with Z such that $\langle 1 \rangle = 1$.

A Criterion for Confinement

Consider the so-called Wilson loop:

$$W(R,T) = \mathrm{tr}\left[\prod_{r=0}^{R-1} U_i(x+rae^{(i)}) \prod_{t=0}^{T-1} U_4(x+Rae^{(i)}+tae^{(4)}) \times \right.$$
$$\left. \prod_{r=1}^{R} U_i^\dagger(x+(R-r)ae^{(i)}+Tae^{(4)}) \prod_{t=1}^{T} U_4^\dagger(x+(T-t)ae^{(4)}) \right]$$

It corresponds to the energy of two static sources of color, separated by distance R, evolving forward through time T.

In a confining theory, one expects the energy to grow with R;
otherwise $V(R) \sim \mathrm{const} - \alpha/R$.

$$\langle W(R,T)\rangle \sim e^{-TV(R)} = \begin{cases} e^{-TR\sigma} & \text{area law, string tension } \sigma \\ e^{-T\mathrm{const}} & \text{perimeter law} \end{cases} .$$

Strong Coupling Expansion

In the pure gauge theory, it is very easy to expand in powers of $\beta \propto g_0^{-2}$.

$$e^{-S} = \prod_{x,\mu,\nu} (1 - \frac{\beta}{2N} P_{\mu\nu} + \ldots)$$

Because $\int dU\, U = 0$, $\langle W(R,T) \rangle$ vanishes at order β^0, β^1, $\ldots$.
The integration over the links in the Wilson loop yields zero.

The only way to get non-zero is to pick up a plaquette term for each link in the loop.

But then there are other links for which $\int dU\, U = 0$.

The first non-zero term in the expansion picks up a factor of β for every plaquette on the planar surface enclosed by $W(R,T)$.

$$\langle W(R,T) \rangle \sim (\mathrm{const} \times \beta)^{RT} \Rightarrow \text{confinement with } \sigma \propto -\ln \beta$$

The Transfer Operator

Before introducing fermions for lattice QCD, we have to tie up a loose end.

In the last lecture, I started to switch back and forth between the path integral and canonical quantum mechanics without taking the (temporal) lattice spacing to zero.

The reason for that is that most lattice actions of interest have a time evolution operator that maps wavefunctions at time t to $t+a$.

It should be called the transfer operator, but more commonly, by an abuse of language, it is called the transfer matrix $\hat{\mathbb{T}}$.

If $\hat{\mathbb{T}}$ is real and positive, we can define a "Hamiltonian"

$$\hat{\mathbb{H}} = -\frac{1}{a}\ln\left(\hat{\mathbb{T}}/\mathbb{T}_0\right)$$

The energies obtained from the exponential fall-off are eigenvalues of $\hat{\mathbb{H}}$.

In quantum mechanics (or in bosonic field theory) it is an integral operator:

$$(\hat{\mathbb{T}}\Psi)(x_{t+1}) = \int dx_t\, e^{-\frac{1}{2}[m(x_{t+1})^2/a+aV(x_{t+1})]} e^{mx_{t+1}x_t/a} e^{-\frac{1}{2}[m(x_t)^2/a+aV(x_t)]} \Psi(x_t)$$

For fermions the construction is more complicated, but similar in spirit.

If the discretization of the time derivative extends over more than one time slice, it may be possible to define a multi-slice transfer operator.

$$\hat{\mathbb{H}} = -\frac{1}{na} \ln\left(\hat{\mathbb{T}}_n/\mathbb{T}_{n0}\right)$$

If $\mathbb{T}$ is real and positive, so is $\mathbb{H}$. Then, with several reasonable assumptions, it is possible to work backwards, from the functional integral, through the transfer operator, to a Hilbert space of states, *i.e.*, to canonical quantum theory. (Called Osterwalder-Schrader reconstruction.)

Not necessarily practical (in the context of numerical calculations) if you need many states to build up some Minkowski object.

Lattice Fermions

In the continuum, fermions (such as the quarks in QCD) have the Lagrangian

$$\mathcal{L}(x) = -\bar{\psi}(x)\,(\not{D} + m_0)\,\psi, \qquad \not{D} = D^\mu \gamma_\mu, \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

We would like to find a discretization of this Lagrangian. We've learned how to put in the gauge fields, so let's focus on the free case.

The simple choices $\partial_\mu \to t_\mu - 1$ or $1 - t_{-\mu}$ are not anti-Hermitian, so the particle and anti-particle parts of ψ would propagate differently.

The simplest anti-Hermitian choice is $(t_\mu - t_{-\mu})/2a$, yielding the naive action

$$\mathcal{L}_{\mathrm{NF}} = -\bar{\psi}(x)\left[\frac{1}{2a}\sum_\mu \gamma_\mu (t_\mu - t_{-\mu}) + m_0\right]\psi(x)$$

Naive Propagator

The propagator that one gets from this Lagrangian is ($t > 0$)

$$G(t,\vec{p}) = \int \frac{dp_4}{2\pi} e^{ip_4t} \frac{a}{i\sum_\mu \gamma_\mu \sin(p_\mu a) + m_0 a}$$

$$= \frac{1}{\sinh(2Ea)} \left[e^{-Et} \left(\gamma_4 \sinh(Ea) - i\sum_i \gamma_i \sin(p_i a) + m_0 a \right) \right.$$

$$\left. + \left(-e^{-Ea}\right)^{t/a} \left(-\gamma_4 \sinh(Ea) + i\sum_i \gamma_i \sin(p_i a) + m_0 a \right) \right]$$

where $\sinh^2(Ea) = \sum_i \sin^2(p_i a) + (m_0 a)^2$.

The first term is desirable; the second term has a peculiar oscillation $(-1)^{t/a}$. Moreover, there are low-lying states for $p_i a \sim 0, (\pi,0,0), (\pi,\pi,0), (\pi,\pi,\pi) \cdots$. $2 \times 8 = 16$ species in all. The Fermion Doubling Problem.

In perturbation theory, 16 species arise from the 16 regions where $\sin(p_\mu a) \sim a$.

Vacuum polarization:

$$a\frac{dg_0^2}{da} = 2\frac{g_0^4}{16\pi^2}\left(\frac{11N}{3} - \frac{2n_f}{3}\right) + O(g_0^6)$$

With naive fermions one find a result of this form but with $n_f = 16n_\psi$.

Anomaly in flavor singlet axial current $A_\mu = \frac{1}{2}\bar{\psi}(T_\mu + T_{-\mu})\gamma_\mu\gamma_5\psi$ from gauging

$$\psi \mapsto e^{i\alpha\gamma_5}\psi, \qquad \bar{\psi} \mapsto \bar{\psi}e^{i\alpha\gamma_5}.$$

In continuum obtain $\partial \cdot A \propto \mathcal{A}$, where $\mathcal{A}$ is the axial anomaly. Now

$$\mathcal{A}_{\mathrm{NF}} = (1 - 4 + 6 - 4 + 1)\mathcal{A} = 0$$

On lattice, symmetries are either exact or explicitly broken.

The Fermion Doubling Problem

The spectrum of naive fermions is the first sign that fermions do not like the lattice.

The Nielsen-Ninomiya Theorem says that there is no *ultra*-local fermion action with the full chiral symmetry, no additional states, and a real, positive transfer matrix.

For a long time “ultra-local” was phrased “local”. Ultra-local means that interactions coupling fields vanish if the fields are farther apart than some fixed distance, of order a few lattice spacings.

“Local” means that they merely fall off exponentially.

There are now fermion discretizations with undoubled spectra and full chiral symmetry. Not ultra-local so no transfer matrix.

Wilson Fermions

To cope with the doubled spectrum, Wilson reasoned as follows.

The particle states should use projection matrices so that some components move forward in time, and others backward.

$$\partial_4\psi(x) \to \left(\frac{1-\gamma_4}{2}\frac{t_4-1}{a} + \frac{1+\gamma_4}{2}\frac{1-t_{-4}}{a}\right)\psi(x)$$

$$\begin{aligned}\bar\psi(x)\gamma_4\partial_4\psi(x) &\to \bar\psi(x)\left(\frac{\gamma_4-1}{2}\frac{t_4-1}{a} + \frac{1+\gamma_4}{2}\frac{1-t_{-4}}{a}\right)\psi(x) \\ &= \bar\psi(x)\left(\gamma_4\frac{t_4-t_{-4}}{2a} - \tfrac{1}{2}a\frac{t_4+t_{-4}-2}{a^2}\right)\psi(x)\end{aligned}$$

Repeat in all directions, leading to

$$\mathcal{L}_{\mathrm{WF}} = \mathcal{L}_{\mathrm{NF}} - \tfrac{1}{2}a\sum_\mu \bar\psi(x)\left(\frac{t_\mu+t_{-\mu}-2}{a^2}\right)\psi(x)$$

Propagator for free Wilson fermions (for $t > 0$)

$$G(t,\vec{p}) = \frac{ae^{-Et}}{2\sinh(Ea)} \frac{\gamma_4 \sinh(Ea) - i\sum_i \gamma_i \sin(p_i a) + m_0 a + \frac{1}{2}a^2\hat{\vec{p}}^2 + 1 - \cosh(Ea)}{1 + m_0 a + \frac{1}{2}a^2\hat{\vec{p}}^2}$$

with

$$\cosh(Ea) = 1 + \frac{1}{2}\frac{\sum_i \sin^2(p_i a) + (m_0 a + \frac{1}{2}a^2\hat{\vec{p}}^2)^2}{1 + m_0 a + \frac{1}{2}a^2\hat{\vec{p}}^2}$$

Now no oscillating state arises, and the energy is low only when $\vec{p}$ is small.

The price paid is sacrificing chiral symmetry $\psi \mapsto e^{i\alpha\gamma_5}\psi$, $\bar{\psi} \mapsto \bar{\psi}e^{i\alpha\gamma_5}$. Both the mass term and the new Wilson term break chiral symmetry explicitly.

$$m_R = Z_m(g_0^2)\left[m_0 + a^{-1}g_0^2 C(g_0^2)\right].$$

On the other hand, the axial anomaly does come out correctly: the Wilson term leads to contributions of the form $a\int d^4k/k^3 \sim aa^{-1} \sim a^0$.

Staggered Fermions

In the early days, Susskind studied a Hamiltonian lattice gauge theory (discrete space, continuous time, Hamiltonian with conjugate momenta). He found a way to formulate the fermions with less doubling.

On a spacetime lattice start with the naive fermion Lagrangian, re-written here

$$\mathcal{L}_{\mathrm{NF}} = -\bar{\psi}(n)\left[\frac{1}{2a}\sum_{\mu}\gamma_{\mu}(T_{\mu} - T_{-\mu}) + m_0\right]\psi(n)$$

with $n \in \mathbb{Z}^4$ dimensionless site labels.

Introduce a unitary similarity transformation

$$\psi(n) = \Omega(n)\Psi(n), \qquad \bar{\psi}(n) = \bar{\Psi}(n)\Omega^{\dagger}(n), \quad \Omega(n) = \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}$$

$$\Omega^{\dagger}(n)\gamma_{\mu}T_{\pm\mu}\Omega(n) = \eta_{\mu}(n)T_{\pm\mu}, \qquad \eta_{\mu}(n) = (-1)^{n_1+\cdots n_{\mu-1}}$$

After the transformation, the Lagrangian is

$$\mathcal{L}_{\mathrm{NF}} = -\sum_{\alpha=1}^{4} \bar{\Psi}_\alpha(n) \left[\frac{1}{2a}\sum_\mu \eta_\mu(n)(T_\mu - T_{-\mu}) + m_0\right] \Psi_\alpha(n)$$

Chiral symmetry remains intact. After the similarity transformation

$$\Omega^\dagger(n)\gamma_5\Omega(n) = \gamma_5\eta_5(n), \qquad \eta_5(n) = (-1)^{n_1+n_2+n_3+n_4} =: \varepsilon(n)$$

$$\Psi(n) \mapsto e^{i\alpha\gamma_5\eta_5(n)}\Psi(n), \qquad \bar{\Psi}(n) \mapsto \bar{\Psi}(n)e^{i\alpha\gamma_5\eta_5(n)}.$$

The chiral transformation rotates even sites ($n_1+n_2+n_3+n_4 \bmod 2 = 0$) one way and odd sites ($n_1+n_2+n_3+n_4 \bmod 2 = 1$) the other, so it is still global.

Written this way, the lattice fermion field has 4 pieces, each with the same Lagrangian; two each with $\gamma_5\Psi = \pm\Psi$. Truncate $\sum_{\alpha=1}^{4}$ to one component.

The Lagrangian for the one component field χ is

$$\mathcal{L}_{\text{stag}} = -\bar{\chi}(n)\left[\frac{1}{2a}\sum_{\mu}\eta_{\mu}(n)(T_{\mu}-T_{-\mu})+m_0\right]\chi(n)$$

with U(1)$\times$U(1) chiral symmetry

$$\chi(n) \mapsto e^{i\theta+i\alpha\eta_5(n)}\chi(n), \qquad \bar{\chi}(n) \mapsto \bar{\chi}(n)e^{-i\theta+i\alpha\eta_5(n)}.$$

This symmetry is enough to forbid an additive counter-term to the bare mass.

Vacuum polarization now behaves as if there are four species. Originally, these were called flavors, in the hope that the four species could be given different masses and correspond to u, d, s, c. Now they are looked at as unphysical and called “tastes”.

The Noether currents corresponding to the U(1)$\times$U(1) chiral symmetry are conserved; the symmetry is exact. We will find another current with the correct anomaly.

Staggered fermions still have the oscillating states; acceptable Hamiltonian $\mathbb{H}$ from a two-slice transfer matrix.

Ginsparg-Wilson Relation

In continuum gauge theories, chiral symmetry (for physical amplitudes) follows essentially from $\{\not{D}, \gamma_5\} = 0$.

It is worth asking whether this condition is (on a lattice) necessary, or merely sufficient.

It is only sufficient; Ginsparg and Wilson derived the necessary condition:

$$D_{\text{lat}}^{-1}\gamma_5 + \gamma_5 D_{\text{lat}}^{-1} = a\gamma_5 \qquad \Rightarrow \qquad \gamma_5 D_{\text{lat}} + D_{\text{lat}}\gamma_5 = aD_{\text{lat}}\gamma_5 D_{\text{lat}}$$

In the form on the left, we see that in correlation functions the violation of chiral symmetry is a local “contact” term, which drops out of the long-distance physics.

Until a few years ago, no solutions (except in free field theory) were known. Now some local, but not ultra-local, solutions have been found.

Numerical Methods for Fermions

The incorporation of fermions into numerical simulations is the most daunting computational problem in lattice QCD.

The Pauli exclusion principle states that two fermions cannot be in the same state.

Therefore, the integration variables in the functional integral are Grassman numbers:

$$\{\psi_a, \psi_b\} = \{\bar{\psi}_a, \psi_b\} = \{\bar{\psi}_a, \bar{\psi}_b\} = 0$$

The integration rule is (α complex, ε & $\bar{\varepsilon}$ Grassman)

$$\int d\psi(\alpha + \bar{\varepsilon}\psi) = -\bar{\varepsilon}, \qquad \int d\bar{\psi}(\alpha + \bar{\psi}\varepsilon) = \varepsilon$$

Invariance under multiplication says

$$\psi = \xi\psi' \qquad \Rightarrow \qquad d\psi = d\pi'/\xi$$

Every action that we introduced to the form

$$S = \sum_{a,b} \bar{\psi}_a M_{ab} \psi_a, \qquad M = M(U)$$

Using the rules of Grassman integration

$$\int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\bar{\psi}M\psi} = \int \mathcal{D}\psi' \mathcal{D}\bar{\psi}' e^{-\bar{\psi}' V^{-1} M V \psi'} = \prod_a [V^{-1} M V]_{aa} = \det M$$

The physical interpretation of $\det M(U)$ is all possible fermions loops in the background of gauge field U.

A numerical simulation the generates gauge fields with weight $\det M \, e^{-S_{\text{gauge}}}$.

This is normal arithmetic, but the computational problem is huge.

M is a $(3 \cdot 4N^4) \times (3 \cdot 4N^4)$ matrix.
(Sparse for naive, staggered, and Wilson, but not GW; omit 4 for staggered.)

Summary of Fermion Methods

Pattern of chiral symmetry breaking for various formulations of lattice fermions.

formulation	$G \to H$	CPU
staggered	$\Gamma_4 \times \mathrm{U}(1) \to \Gamma_4$	fast: $m_q > 0.2 m_s$, but $n_f = 4$
Wilson	$\mathrm{SU}(n_f) \to \mathrm{SU}(n_f)$	slower: $m_q > 0.5 m_s$
G-W	$\mathrm{SU}(n_f) \times \mathrm{SU}(n_f) \to \mathrm{SU}(n_f)$	very slow: M not sparse
continuum QCD	$\mathrm{SU}(n_f) \times \mathrm{SU}(n_f) \to \mathrm{SU}(n_f)$	

For all methods, the computation of $\det M$ (or changes in $\det M$) gets slower and slower as the quark mass decreases (ratio of eigenvalues).